

# OPTIMAL ESTIMATES FOR THE GRADIENT OF HARMONIC FUNCTIONS IN THE UNIT DISK

DAVID KALAJ AND MARIJAN MARKOVIĆ

ABSTRACT. Let  $\mathbf{U}$  be the unit disk,  $p \geq 1$  and let  $h^p(\mathbf{U})$  be the Hardy space of complex harmonic functions. We find the sharp constants  $C_p$  and the sharp functions  $C_p = C_p(z)$  in the inequality

$$|Dw(z)| \leq C_p(1 - |z|^2)^{-1-1/p} \|w\|_{h^p(\mathbf{U})}, w \in h^p(\mathbf{U}), z \in \mathbf{U},$$

in terms of Gauss hypergeometric and Euler functions. This generalizes some results of Colonna related to the Bloch constant of harmonic mappings of the unit disk into itself and improves some classical inequalities by Macintyre and Rogosinski.

## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

A harmonic function  $w$  defined in the unit ball  $\mathbf{B}^n$  belongs to the harmonic Hardy class  $h^p = h^p(\mathbf{B}^n)$ ,  $1 \leq p < \infty$  if the following growth condition is satisfied

$$(1.1) \quad \|w\|_{h^p} := \left( \sup_{0 < r < 1} \int_S |w(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} < \infty$$

where  $S = S^{n-1}$  is the unit sphere in  $R^n$  and  $\sigma$  is the unique normalized rotation invariant Borel measure on  $S$ . The space  $h^\infty(\mathbf{B}^n)$  contains bounded harmonic functions.

It turns out that if  $w \in h^p(\mathbf{B}^n)$ , then there exists the finite radial limit

$$\lim_{r \rightarrow 1^-} w(r\zeta) = f(\zeta) \text{ (a.e. on } S)$$

and the boundary function  $f(\zeta)$  belong to the space  $L^p(S)$  of  $p$ -integrable functions on the sphere.

It is well known that harmonic functions from Hardy class can be represented as Poisson integral

$$u(x) = \int_S P(x, \zeta) d\mu(\zeta), x \in B^n$$

where

$$P(x, \zeta) = \frac{1 - |x|^2}{|x - \zeta|^n}, x \in B^n, \zeta \in S$$

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is Poisson kernel and  $\mu$  is complex Borel measure. In the case  $p > 1$  this measure is absolutely continuous with respect to  $\sigma$  and  $d\mu(\zeta) = f(\zeta)d\sigma$ . Moreover

$$\|w\|_{h^p} = \|\mu\|$$

and for  $1 < p \leq \infty$  we have

$$(1.2) \quad \|w\|_{h^p} = \|\mu\| = \|f\|_p.$$

where we denote by  $\|\mu\|$  total variation of the measure  $\mu$ .

For previous facts we refer to the book [1, Chapter 6].

For  $n = 2$  we use the classical notation  $\mathbf{U}$  and  $\mathbf{T}$  to denote the unit disk in the complex plane  $\mathbf{C}$  and its boundary.

Let  $L^p(\mathbf{R}^n)$  be the space of Lebesgue integrable functions defined in  $\mathbf{R}^n$  with the norm

$$\|f\|_p = \left( \int_{\mathbf{R}^n} |f(x')|^p dx' \right)^{1/p}.$$

Let  $\omega_n$  be the area of the unit sphere in  $\mathbf{R}^n$ . Let in addition  $h^p(\mathbf{R}_+^n)$  be the Hardy space of real harmonic functions in  $\mathbf{R}_+^n$ , which can be represented as the Poisson integral

$$u(x) = \frac{2}{\omega_n} \int_{\mathbf{R}^n} \frac{x_n}{|y - x|^n} u(y') dy',$$

with boundary values in  $L^p(\mathbf{R}^{n-1})$ , where  $y = (y', 0)$ ,  $y' \in \mathbf{R}^{n-1}$ .

In the recent paper [7] Maz'ya and Kresin studied point-wise estimates of the gradient of real harmonic function  $u$  under the assumptions that the boundary values belong to  $L^p$ . They obtained the following result

$$|\nabla u(x)| \leq C_p x_n^{(1-n-p)/p} \|u\|_p$$

where  $C_p$  is a constant depending only on  $p$  and  $n$ . For  $p = 1$ ,  $p = 2$  and  $p = \infty$  the constant  $C_p$  is concretized and it is shown the sharpness of the result. After that, in [8], they obtained similar results for the unit ball, but for  $p = 1$  and  $p = 2$  only. Precisely, they obtain some integral representation for the sharp constant  $K_p(x, l)$  in the inequality

$$|\langle \nabla u(x), l \rangle| \leq K_p(x, l) \|u\|_p, \quad 1 \leq p \leq \infty$$

and the sharp constant  $K_p(x)$  in

$$|\nabla u(x)| \leq K_p(x) \|u\|_p$$

is concretized for  $p = 1, 2$  and  $x$  arbitrary and for  $x = 0$  and all  $p$ .

Notice that, for  $n = 2$  the results concerning the upper half-plane  $\mathbf{H}$  cannot be directly translated to the unit disk and vice-versa. Although the unit disk  $\mathbf{U}$  and the upper half-plane  $\mathbf{H}$  can be mapped to one-another by means of Möbius transformations, they are not interchangeable as domains for Hardy spaces. Contributing to this difference is the fact that the unit circle has finite (one-dimensional) Lebesgue measure while the real line does not.

A complex harmonic function  $w$  in a region  $D$  can be expressed as  $w = u + iv$  where  $u$  and  $v$  are real harmonic functions in  $D$ . For a complex harmonic function we will use sometimes the abbreviation a harmonic mapping. If  $D$  is simply-connected, then there are two analytic functions  $h$  and  $k$  defined on  $D$  such that  $w = g + \bar{h}$ . For a complex harmonic function  $w = g + \bar{h} = u + iv$ , denote by  $Dw(z)$  the formal differential matrix  $Dw(z) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ . Its norm is given by

$$|Dw| := \max\{|Dw(z)l| : |l| = 1\}.$$

Then

$$(1.3) \quad |Dw(z)| = |g'(z)| + |h'(z)|.$$

Let  $w$  be a harmonic function satisfying the Lipschitz condition, when regarded as a function from the hyperbolic unit disk into the complex plane  $\mathbf{C}$  endowed with the Euclidean distance. The function  $w$  is called *Bloch* with the *Bloch constants*

$$\beta_w = \sup_{z \neq z'} \frac{|w(z) - w(z')|}{d_h(z, z')}.$$

Here  $d_h$  is defined by

$$\tanh \frac{d_h(z, z')}{2} = \frac{|z - z'|}{|1 - z\bar{z}'|}.$$

It can be proved that

$$(1.4) \quad \beta_w = \sup_{z \in \mathbf{U}} (1 - |z|^2) |Dw(z)|.$$

We refer to [2, Theorem 1] for the proof of (1.4). In the same paper Colonna proved that, if  $w$  is a harmonic mapping of the unit disk into itself, then there hold the following sharp inequality

$$(1.5) \quad \beta_w \leq \frac{4}{\pi}.$$

See also the book of Pavlović [12, p. 53, 54] for a related problem.

An estimates similar to (1.5) for magnitudes of derivatives of bounded harmonic functions in the unit ball in  $\mathbf{R}^3$  is obtained by Khavinson in [4].

Together with the Bloch constants, for a harmonic mapping of the unit disk onto itself consider the *hyperbolic Lipschitz constant* defined by

$$\beta_w^{\text{hyp}} := \sup_{z \neq z'} \frac{d_h(w(z), w(z'))}{d_h(z, z')}.$$

Since  $|dz| \leq |dz|/(1 - |z|^2)$ , it follows that for  $z, w \in \mathbf{U}$  we have  $d(z, w) \leq d_h(z, w)$ . Thus

$$\beta_w \leq \beta_w^{\text{hyp}}.$$

It follows by Schwarz-Pick lemma that, if  $w$  is an analytic function then

$$\beta_w^{\text{hyp}} \leq 1,$$

and the equality is attained for Möbius selfmappings of the unit disk. Very recently it is proved in [5] that, for real harmonic mappings of the unit disk onto itself there hold the following sharp inequality

$$(1.6) \quad |\nabla w| \leq \frac{4}{\pi} \frac{1 - |w(z)|^2}{1 - |z|^2},$$

and therefore  $\beta_w^{\text{hyp}} \leq \frac{4}{\pi}$  extending thus Colonna result for real harmonic mappings. However if we drop the assumption that  $w$  is real, then  $\beta_w^{\text{hyp}}$  can be infinite. The inequality (1.6) can be considered as a real-part theorem for an analytic function. More than one approach can be found in the book [9].

In this paper we prove the following results for the unit disk which are analogous to the results of Maz'ya & Kresin and extend the results of Colonna by proving the following theorems.

Since the case  $p = 1$  is well-known, we will assume in the sequel that  $p > 1$ .

**Theorem 1.1** (Main theorem). *Let  $p > 1$  and let  $q$  be its conjugate. Let  $w \in h^p$  be a complex harmonic function defined in the unit disk and let  $z \neq 0$ . Define  $\mathbf{n} = \frac{\bar{z}}{|z|}$ , and  $\mathbf{t} = i \frac{z}{|z|}$ .*

*a) We have the following sharp inequalities*

$$(1.7) \quad |Dw(z)e^{i\tau}| \leq C_p(z, e^{i\tau})(1 - r^2)^{-1/p-1} \|w\|_{h^p},$$

$$(1.8) \quad |Dw(z)| \leq C_p(z)(1 - r^2)^{-1/p-1} \|w\|_{h^p},$$

where  $z = re^{i\alpha}$ ,

$$C_p(z, e^{i\tau}) = \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{|\cos(s + \tau - \alpha)|^q}{(1 + r^2 - 2r \cos s)^{1-q}} ds \right)^{1/q}$$

and

$$(1.9) \quad C_p(z) = \begin{cases} C_p(z, \mathbf{n}), & \text{if } p < 2; \\ C_p(z, \mathbf{t}), & \text{if } p \geq 2. \end{cases}$$

Moreover

$$(1.10) \quad \begin{cases} C_p(z, \mathbf{t}) \leq C_p(z, e^{i\tau}) \leq C_p(z, \mathbf{n}), & \text{if } p < 2; \\ C_p(z, \mathbf{n}) \leq C_p(z, e^{i\tau}) \leq C_p(z, \mathbf{t}), & \text{if } p \geq 2. \end{cases}$$

*b) For  $p \geq 2$  the function  $C_p(z)$  can be expressed as*

$$(1.11) \quad C_p(z) = \frac{2^{1/q}}{\pi} \left( B\left(\frac{1+q}{2}, \frac{1}{2}\right) F\left(1 - \frac{3q}{2}, 1 - q; 1 + \frac{q}{2}; r^2\right) \right)^{1/q},$$

where  $B$  is the beta function and  $F$  is the Gauss hypergeometric function.

c) Finally

$$C_p := \sup_{z \in \mathbf{U}} C_p(z) = \begin{cases} \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{|\cos s|^q}{(2-2\cos s)^{1-q}} ds \right)^{1/q}, & \text{if } 1 < p < 2; \\ \frac{1}{\pi} \left( \int_{-\pi}^{\pi} \frac{|\sin s|^q}{(2-2\cos s)^{1-q}} ds \right)^{1/q}, & \text{if } p \geq 2. \end{cases}$$

The constant  $C_p$  is optimal for real harmonic functions as well.

**Theorem 1.2.** Let  $p > 1$  and let  $w \in h^p$ , be a complex harmonic function defined in the unit disk. Then we have the following sharp inequalities

$$|\partial w(z)|, |\bar{\partial} w(z)| \leq c_p(z)(1 - |z|^2)^{-1/p-1} \|w\|_{h^p},$$

and

$$|\partial w(z)|, |\bar{\partial} w(z)| \leq c_p(1 - |z|^2)^{-1/p-1} \|w\|_{h^p},$$

where

$$(1.12) \quad c_p(z) = (2\pi)^{1/q-1} (F(1-q, 1-q; 1; r^2))^{1/q}$$

and

$$(1.13) \quad c_p = 2^{\frac{-1+q}{q}} \pi^{-1+\frac{1}{2q}} \left( \frac{\Gamma(-1/2+q)}{\Gamma(q)} \right)^{1/q}.$$

*Remark 1.3.* a) In particular, if in Theorem 1.1 we take  $p = 2$ , then we have the following estimate

$$(1.14) \quad |\nabla w(z)| \leq \frac{1}{\sqrt{\pi}} \frac{(1 + |z|^2)^{1/2}}{(1 - |z|^2)^{3/2}} \|w\|_{h^2}.$$

If we assume  $w$  is a real harmonic function, i.e.  $w = g + \bar{g}$ , where  $g$  is an analytic function, then this estimate is equivalent to the real part theorem

$$(1.15) \quad |g'(z)| \leq \frac{1}{\sqrt{\pi}} \frac{(1 + |z|^2)^{1/2}}{(1 - |z|^2)^{3/2}} \|\Re g\|_{h^2}.$$

For the proof of (1.15) we refer to [9, pp. 87, 88]. See also a higher dimensional generalization of (1.14) by Maz'ya and Kresin in the recent paper [8, Corollary 3] for  $n \geq 2$ . Also the relation (1.14) for real  $w$  can be deduced from work of Macintyre and Rogosinski for analytic functions, see [10, p. 301].

b) On the other hand if take  $p = \infty$ , then  $C_p = \frac{4}{\pi}$  and therefore the relation (1.8) coincides with the result of Colonna. While for real  $w$ , it is a real part theorem ([4]) which can be expressed as

$$(1.16) \quad |g'(z)| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2} \|\Re g\|_{\infty}.$$

c) Notice that  $c_p < C_p < 2c_p$ , if  $p > 1$ , and  $C_1 = 2c_1 = \frac{4}{\pi}$ . On the other hand  $c_{\infty} = 1$  coincides with the constant of the Schwarz lemma for analytic functions. Notice also this interesting fact, the minimum of constants  $C_p$  is achieved for  $p = 2$  and is equal to  $C_2 = \sqrt{2/\pi}$ . The graphs of functions  $C_p$  and  $c_p$ ,  $1 \leq p \leq 20$  are shown in Figure 1 and Figure 2.

d) From Theorem 1.1 we find out that, the Khavinson hypothesis (see [8]) is not true for  $n = 2$  and  $2 < p < \infty$ . Namely the maximum of the absolute value of the directional derivative of a harmonic function with a fixed  $L^p$ -norm of its boundary values is attained at the radial direction for  $p \leq 2$  and at the tangential direction for  $2 < p < \infty$ .

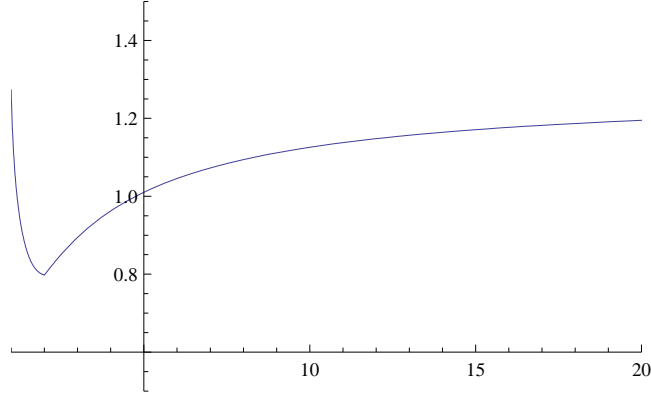


FIGURE 1. The graph of  $C_p$ .

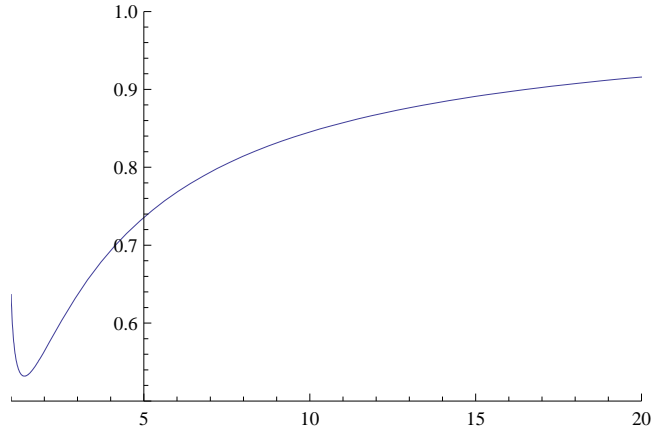


FIGURE 2. The graph of  $c_p$ .

In the classical paper [10, (8.3.8)] of Macintyre and Rogosinski they obtained the inequality

$$(1.17) \quad |f'(z)| \leq \left(1 + \frac{r^2}{(p-1)^2}\right)^{1/q} (1 - |z|^2)^{-1-1/p} \|f\|_{H^p}.$$

In the following direct corollary of Theorem 1.2 we improve the inequality (1.17) by proving

**Corollary 1.4.** *Let  $w = f(z)$  be an analytic function from the Hardy class  $H^p(\mathbf{U})$ . Then there hold the following inequality*

$$(1.18) \quad |f'(z)| \leq c_p(z)(1 - |z|^2)^{-1-1/p} \|f\|_{H^p},$$

where  $c_p(z)$  is defined in (1.12).

*Remark 1.5.* Corollary 1.4 is an improvement of corresponding inequality [10, (8.3.8)] because

$$(2\pi)^{1-q} F(1-q, 1-q; 1; r^2) < 1 + \frac{r^2}{(p-1)^2}$$

for all  $q > 1$ . It is not known by the authors if all functions  $c_p(z)$  of Corollary 1.4 are sharp, however the function  $c_2(z) = \frac{\sqrt{1+|z|^2}}{\sqrt{2\pi}}$  is sharp, because (1.18) coincides with the sharp inequality [10, p. 301, eq. (7.2.1)] for  $p = 2$ . On the other hand the power  $-1 - 1/p$  is optimal see e.g. Garnett [3, p. 86]. The paper [10] contains some sharp estimates  $|f^{(k)}(z)| \leq c_p \|f\|_p$  for  $f \in H^p(\mathbf{U})$  and  $k \geq 1$  but  $p$  depends on  $k$  and it seems that if  $k = 1$  then  $p$  can be only 1 or 2.

## 2. PROOFS

We need the following lemmas

**Lemma 2.1.** *Let  $a_q(t)$ ,  $t \in [0, 2\pi]$ ,  $q \geq 1$ ,  $0 \leq r \leq 1$  be a function defined by*

$$a_q(t) = \int_{-\pi}^{\pi} |\cos(s-t)|^q |r - e^{is}|^{2q-2} ds.$$

Then

$$\max_{0 \leq t \leq 2\pi} a_q(t) = \begin{cases} a_q(\frac{\pi}{2}), & \text{if } q \leq 2; \\ a_q(0), & \text{if } q > 2. \end{cases}$$

and

$$\min_{0 \leq t \leq 2\pi} a_q(t) = \begin{cases} a_q(0), & \text{if } q \leq 2; \\ a_q(\frac{\pi}{2}), & \text{if } q > 2. \end{cases}$$

*Proof.* Since  $q = 1$  is trivial let  $q > 1$  and

$$a(t) := a_q(t) = \int_{-\pi}^{\pi} |\cos(t-s)|^q (1 + r^2 - 2r \cos s)^{q-1} ds.$$

Note that  $a$  is  $\pi$ -periodic. Because sub-integral expression is  $2\pi$ -periodic with respect to  $s$  we obtain

$$a(t) = \int_0^{2\pi} |\cos s|^q (1 + r^2 - 2r \cos(t+s))^{q-1} ds,$$

and therefore

$$a'(t) = 2(q-1)r \int_0^{2\pi} |\cos s|^q \sin(t+s) (1 + r^2 - 2r \cos(t+s))^{q-2} ds.$$

Again by using the periodicity of sub-integral expression

$$a'(t) = 2(q-1)r \int_0^{2\pi} |\cos(t-s)|^q \sin s (1+r^2-2r\cos s)^{q-2} ds.$$

Next we need some transformations

$$\begin{aligned} a'(t) &= 2(q-1)r \int_0^\pi |\cos(t-s)|^q \sin s (1+r^2-2r\cos s)^{q-2} ds \\ &\quad + 2(q-1)r \int_\pi^{2\pi} |\cos(t-s-\pi)|^q \sin(s+\pi) (1+r^2-2r\cos(s+\pi))^{q-2} ds \\ &= 2(q-1)r \int_0^\pi |\cos(t-s)|^q \sin s Q(r, s-\pi/2) ds \\ &= 2(q-1)r \int_{-\pi/2}^{\pi/2} |\sin(t-s)|^q \cos s Q(r, s) ds, \end{aligned}$$

where

$$Q(r, s) = (1+r^2+2r\sin s)^{q-2} - (1+r^2-2r\sin s)^{q-2}.$$

Thus the derivative is

$$a'(t) = 2r(q-1) \int_{-\pi/2}^{\pi/2} h(t, s) \cos s ds,$$

where

$$h(t, s) = |\sin(t-s)|^q Q(r, s).$$

Also  $a'(t)$  is  $\pi$ -periodic and

$$a'(0) = a'(\pi/2) = 0.$$

Further

$$h(t, s) + h(t, -s) = (|\sin(t-s)|^q - |\sin(t+s)|^q) Q(r, s).$$

If  $1 < q < 2$ , then for  $0 < t < \pi/2$  we have

$$h(t, s) + h(t, -s) > 0, \quad 0 < s < \pi/2$$

and  $\pi/2 < t < \pi$

$$h(t, s) + h(t, -s) > 0, \quad 0 < s < \pi/2.$$

We claim that

$$a'(t) = 2r(q-1) \int_0^{\pi/2} (h(t, s) + h(t, -s)) \cos s ds > 0, \quad 0 < t < \pi/2$$

and

$$a'(t) = 2r(q-1) \int_0^{\pi/2} (h(t, s) + h(t, -s)) \cos s ds < 0, \quad \pi/2 < t < \pi.$$

It means that the minimum of  $a$  is achieved in 0 and the maximum in  $\frac{\pi}{2}$ .

Similarly can be treated the case  $q > 2$ . For  $q = 2$  the function  $a(t)$  is a constant. The proof of Lemma 2.1 is completed.  $\square$



**Lemma 2.2.** *Let  $\lambda \geq 0$ ,  $0 \leq r \leq 1$  and  $q \geq 1$ . For all  $t$  there exists  $t' \in [0, 2\pi]$  such that*

$$\int_0^{2\pi} |\cos(s-t)|^\lambda |r - e^{is}|^{2q-2} ds \leq \int_0^{2\pi} |\cos(s-t')|^\lambda |1 - e^{is}|^{2q-2} ds.$$

*Proof.* In order to prove Lemma 2.2, we need the following proposition.

**Proposition 2.3.** [6, Lemma 3.2] *Let  $\mathbf{U} \subset \mathbf{C}$  be the open unit disk and  $(A, \mu)$  be a measured space with  $\mu(A) < \infty$ . Let  $f(z, \omega)$  be a holomorphic function for  $z \in \mathbf{U}$  and measurable for  $\omega \in A$ . Let  $b > 0$  and assume in addition that, there exists an integrable function  $\chi \in L^{\max\{b, 2\}}(A, d\mu)$  such that*

$$(2.1) \quad |f(0, \omega)| + |f'(z, \omega)| \leq \chi(\omega),$$

*for  $(z, \omega) \in \mathbf{U} \times A$ , where by  $f'(z, \omega)$  we mean the complex derivative of  $f$  with respect to  $z$ . Then the function*

$$\phi(z) = \log \int_A |f(z, \omega)|^b d\mu(\omega)$$

*is subharmonic in  $\mathbf{U}$ .*

**Corollary 2.4.** *Assume together with the assumptions of the previous proposition that  $z \rightarrow f(z, \omega)$  is continuous up to the boundary  $\mathbf{T}$ . Then we have the following inequality*

$$\phi(z) \leq \max_{\tau \in [0, 2\pi]} \phi(e^{i\tau}) = \phi(e^{i\tau'}).$$

In order to apply Corollary 2.4, we take

$$d\mu(s) = |\cos(s-t)|^\lambda ds, \quad f(z, s) = z - e^{is} \text{ and } b = 2q - 2$$

and observe that

$$\begin{aligned} \max_{\tau} \int_0^{2\pi} |\cos(s-t)|^\lambda |e^{i\tau} - e^{is}|^{2q-2} ds \\ = \int_0^{2\pi} |\cos(s-t)|^\lambda |e^{i\tau'} - e^{is}|^{2q-2} ds \\ = \int_0^{2\pi} |\cos(s-t')|^\lambda |1 - e^{is}|^{2q-2} ds. \end{aligned}$$

This finishes the proof of Lemma 2.2. □

The Poisson kernel for the disc can be expressed as

$$P(z, e^\theta) = \frac{1 - |z|^2}{|z - e^{i\theta}|^2} = - \left( 1 + \frac{e^{-i\theta}}{\bar{z} - e^{-i\theta}} + \frac{e^{i\theta}}{z - e^{i\theta}} \right).$$

Then we have

$$\text{grad}(P) = (P_x, P_y) = P_x + iP_y = 2\bar{\partial}P = \frac{2e^{-i\theta}}{(\bar{z} - e^{-i\theta})^2},$$

$$\partial P = \frac{e^{i\theta}}{(z - e^{i\theta})^2}$$

and

$$\bar{\partial} P = \frac{e^{-i\theta}}{(\bar{z} - e^{-i\theta})^2}.$$

*Proof of Theorem 1.1.* a) Let  $l = e^{i\tau}$ . Then for  $p > 1$

$$\begin{aligned} Dw(z)l &= \frac{1}{2\pi} \int_0^{2\pi} \langle \text{grad}(P), l \rangle f(e^{i\theta}) d\theta \\ (2.2) \quad &= \frac{1}{\pi} \int_0^{2\pi} \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} f(e^{i\theta}) d\theta. \end{aligned}$$

We apply (2.2) and Hölder inequality in order to obtain

$$|Dw(z)l| \leq \frac{1}{\pi} \left( \int_0^{2\pi} \left| \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} \right|^q d\theta \right)^{1/q} \left( \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{1/p}.$$

We should consider the integral

$$I_q = \int_0^{2\pi} \left| \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} \right|^q d\theta.$$

First of all

$$I_q = \int_0^{2\pi} \left| \Re \frac{e^{i(\theta+\tau)}}{(z - e^{i\theta})^2} \right|^q d\theta = \int_0^{2\pi} \left| \Re \frac{e^{i(\theta+\tau-\alpha)}}{(r - e^{i\theta})^2} \right|^q d\theta.$$

Take the substitution

$$e^{i\theta} = \frac{r - e^{is}}{1 - re^{is}}.$$

Then

$$de^{i\theta} = \frac{1 - r^2}{(1 - re^{is})^2} de^{is},$$

and thus

$$\begin{aligned} d\theta &= \frac{1 - r^2}{(1 - re^{is})^2} \frac{e^{is}}{e^{i\theta}} ds \\ &= e^{is} \frac{1 - r^2}{(1 - re^{is})^2} \frac{1 - re^{is}}{r - e^{is}} ds \\ &= \frac{1 - r^2}{1 + r^2 - 2r \cos s} ds. \end{aligned}$$

On the other hand, we easily find that

$$\Re \frac{e^{i(\theta+\tau-\alpha)}}{(r - e^{i\theta})^2} = \frac{(1 + r^2 - 2r \cos s) \cos(s + \tau - \alpha)}{(1 - r^2)^2}.$$

Therefore, finally we have the relation

$$(2.3) \quad \int_0^{2\pi} \left| \Re \frac{e^{-i(\theta+\tau)}}{(\bar{z} - e^{-i\theta})^2} \right|^q d\theta = (1 - |z|^2)^{1-2q} \int_{-\pi}^{\pi} \frac{|\cos(s + \tau - \alpha)|^q}{(1 + r^2 - 2r \cos s)^{1-q}} ds,$$

which together with first relation give

$$|Dw(z)l| \leq C_p(z, l)(1 - |z|^2)^{-1-1/p} \|w\|_{h^p}.$$

Now by using Lemma 2.1 we conclude that

$$C_p(z) = \begin{cases} C_p(z, \mathbf{n}), & \text{if } p < 2; \\ C_p(z, \mathbf{t}), & \text{if } p \geq 2, \end{cases}$$

which coincides with (1.9). This implies (1.8). Lemma 2.1 implies at once (1.10).

b) By using the following formula

$$(2.4) \quad \int_0^{\pi} \frac{\sin^{\mu-1} t}{(1 + r^2 - 2r \cos t)^{\nu}} dt = B\left(\frac{\mu}{2}, \frac{1}{2}\right) F\left(\nu, \nu + \frac{1-\mu}{2}; \frac{1+\mu}{2}, r^2\right)$$

(see, e.g., Prudnikov, Brychkov and Marichev [11, 2.5.16(43)]), where  $B(u, v)$  is the Beta-function, and  $F(a, b; c; x)$  is the hypergeometric Gauss function, for  $\mu = q+1$  and  $\nu = 1-q$ , because  $|\cos(s + \tau - \alpha)|^q = |\sin s|^q$ , for  $\tau = \alpha + \frac{\pi}{2}$ , we obtain (1.11).

c) By using both Lemma 2.2 and Lemma 2.1 we obtain:

$$C_p(z, l) \leq C_p(1, l') \leq C_p$$

for some  $l', |l'| = 1$  and we have second conclusion of main theorem.

Let us now show that the constant  $C_p$  is sharp. We will show the sharpness of the result for  $p \leq 2$ . A similar analysis works for  $p > 2$ . Let  $0 < \rho < 1$  and take

$$e^{is} = \frac{\rho - e^{it}}{1 - \rho e^{it}},$$

i.e.

$$e^{it} = \frac{\rho - e^{is}}{1 - \rho e^{is}}.$$

Define

$$f_{\rho}(e^{it}) = (1 - \rho^2)^{-1/p} |\cos s (1 - \cos s)|^{q-1} \operatorname{sign}(\cos s).$$

And take

$$w_{\rho} = P[f_{\rho}].$$

Then

$$dt = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos s} ds,$$

$$\Re \frac{e^{it}}{(r - e^{it})^2} = \frac{(1 + r^2 - 2r \cos s) \cos(s)}{(1 - r^2)^2},$$

and

$$\begin{aligned} \int_0^{2\pi} |f_\rho(e^{it})|^p dt &= \int_0^{2\pi} |f_\rho(e^{it})|^p \frac{1}{1 + \rho^2 - 2\rho \cos s} ds \\ &= \int_0^{2\pi} |\cos s(1 - \cos s)|^q \frac{1}{1 + \rho^2 - 2\rho \cos s} ds. \end{aligned}$$

Thus

$$(2.5) \quad \lim_{\rho \rightarrow 1} \|f_\rho\|_p^p = \int_0^{2\pi} |f_\rho(e^{it})|^p dt = \frac{\pi^q}{2^q} C_p^q.$$

By taking  $r = \rho$ , we obtain

$$\begin{aligned} (1 - \rho^2)^{1+1/p} |Dw_\rho(\rho)1| &= \frac{(1 - \rho^2)^{1+1/p}}{\pi} \int_0^{2\pi} \Re \frac{e^{it}}{(\rho - e^{i\theta})^2} f_\rho(e^{it}) dt \\ &= \frac{(1 - \rho^2)^{1+1/p}}{\pi} \int_0^{2\pi} \frac{(1 + \rho^2 - 2\rho \cos s) \cos(s)}{(1 - \rho^2)^2} (1 - \rho^2)^{-1/p} \\ &\quad \times |\cos s(1 - \cos s)|^{q-1} \operatorname{sign}(\cos s) \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos s} ds \\ &= \frac{1}{\pi} \int_0^{2\pi} |\cos s|^q (1 - \cos s)^{q-1} ds \\ &= \frac{\pi^{q-1}}{2^{q-1}} C_p^q \end{aligned}$$

From (2.5) it follows that

$$\lim_{\rho \rightarrow 1} \frac{(1 - \rho^2)^{1+1/p} |Dw_\rho(\rho)1|}{\|f_\rho\|_p} = C_p.$$

This shows that the constant  $C_p$  is sharp.  $\square$

*Proof of Theorem 1.2.* First of all

$$\partial w = \int_0^{2\pi} \frac{e^{i\theta}}{(z - e^{i\theta})^2} f(e^{i\theta}) \frac{d\theta}{2\pi}.$$

By applying Hölder inequality we have

$$\begin{aligned} |\partial w| &\leq \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{1}{|z - e^{i\theta}|^{2q}} d\theta \right)^{1/q} \left( \int_0^{2\pi} |f(e^{i\theta})|^p dt \right)^{1/p} \\ &= (1 - |z|^2)^{1/q-2} \frac{1}{2\pi} \left( \int_0^{2\pi} \frac{(1 - |z|^2)^{2q-1}}{|z - e^{i\theta}|^{2q}} d\theta \right)^{1/q} \left( \int_0^{2\pi} |f(e^{i\theta})|^p dt \right)^{1/p}. \end{aligned}$$

It remains to estimate the integral

$$J_q = \int_0^{2\pi} \frac{(1 - |z|^2)^{2q-1}}{|z - e^{i\theta}|^{2q}} d\theta = \int_0^{2\pi} \frac{(1 - r^2)^{2q-1}}{|r - e^{i\theta}|^{2q}} d\theta.$$

By making use again of the change

$$e^{i\theta} = \frac{r - e^{is}}{1 - re^{is}},$$

we obtain

$$d\theta = \frac{1-r^2}{|1-re^{is}|^2} ds$$

and

$$r - e^{i\theta} = \frac{(1-r^2)e^{is}}{1-re^{is}}.$$

Therefore by using Lemma 2.2 for  $\lambda = 0$  we obtain

$$\begin{aligned} J_q &= \int_0^{2\pi} \frac{(1-r^2)^{2q-1}}{|r-e^{i\theta}|^{2q}} d\theta = (1-r^2)^{1-q} \int_0^{2\pi} |1-re^{is}|^{2q-2} ds \\ &= (1-r^2)^{1-q} \int_0^{2\pi} |1+r^2-2r\cos s|^{q-1} ds \\ &\leq 2^{q-1} (1-r^2)^{1-q} \int_0^{2\pi} |1-\cos s|^{q-1} ds. \end{aligned}$$

Thus

$$|\partial w| \leq c_p (1-|z|^2)^{-1-1/p} \|f\|_{L^p(\mathbf{T})},$$

where

$$c_p = 2^{\frac{-1+q}{q}} \pi^{-1+\frac{1}{2q}} \left( \frac{\Gamma(-1/2+q)}{\Gamma(q)} \right)^{1/q}.$$

This proves (1.13). By formula (2.4) for  $\mu = 1$ ,  $\nu = 1 - q$  we have

$$\begin{aligned} \int_0^{2\pi} |1+r^2-2r\cos s|^{q-1} ds &= 2 \int_0^\pi |1+r^2-2r\cos s|^{q-1} ds \\ &= 2\pi F(1-q, 1-q; 1, r^2). \end{aligned}$$

This implies (1.12). The sharpness of constant  $c_p$  can be verified by taking

$$f_\rho^\pm(e^{it}) = (1-\rho^2)^{-1/p} |\cos s(1-\cos s)|^{q-1} e^{\pm is}$$

and following the proof of sharpness of  $C_p$ .  $\square$

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UNIVERSITY OF MONTENEGRO, FACULTY OF NATURAL SCIENCES AND MATHEMATICS,  
CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO

*E-mail address:* davidk@ac.me

UNIVERSITY OF MONTENEGRO, FACULTY OF NATURAL SCIENCES AND MATHEMATICS,  
CETINJSKI PUT B.B. 81000 PODGORICA, MONTENEGRO

*E-mail address:* marijanmarkovic@gmail.com